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RECOGNIZING GRAPHIC MATROIDS

by

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There is no polynomially bounded algorithm to test if a matroid (presented by an "independence oracle") is binary. However, there is one to test graphicness. Finding this extends work of previous authors, who have given algorithms to test binary matroids for graphicness. Our main tool is a new result that if M' is the polygon matroid of a graph G, and M is a different matroid on $E(G)$ with the same rank, then there is a vertex of G whose star is not a cocircuit of M.

1. The Theorem

We shall assume familiarity with matroid theory — for an introduction, see Welsh [7]. We begin with some definitions. $E(M)$ is the set of elements of the matroid M. The rank of M is denoted by $r(M)$. For $e\in E(M)$; $M\setminus e$ is the matroid with element set $E(M)-\{e\}$, of which a subset is independent if and only if it is independent in M. The dual matroid of M is denoted by M^*, and M/e abbreviates $(M^*\setminus e)^*$. The prefix "co-" dualizes a term; thus, coloop, cocircuit. The polygon matroid of the graph G is denoted by $\mathcal{M}(G)$. Graphs may have loops or multiple edges. When $G=(V, E)$ is a graph and $X\subseteq V$, $\partial(X)=\partial_G(X)$ denotes the set of edges with one end in X and the other in $V-X$. Such a subset of E is called a *cut*. Thus cuts contain no loops. A *star* is a cut of the form $\partial(\{v\})$ where $v\in V$. An *isthmus* is an edge $e\in E$ such that $\{e\}$ is a cut. For $e\in E$, $G\setminus e$ and G/e denote the graphs obtained from G by deleting and contracting e respectively.

Our main theorem is the following.

Theorem. *Let $G=(V, E)$ be a graph and let M be a matroid on E. Then $M=\mathcal{M}(G)$ if and only if both the following conditions hold:*

(i) $r(M)\leqq r(\mathcal{M}(G))$

(ii) *every star of G is expressible as the union of cocircuits of M.*

Proof. Necessity is clear. To prove sufficiency we proceed by induction on $|E(G)|$. Let M satisfy (i) and (ii).

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We assume, for a contradiction, that $M \neq \mathcal{M}(G)$. Then there exists $X \subseteq E$, dependent in one of M, $\mathcal{M}(G)$ and independent in the other. Choose such an X, minimal. Then X is a circuit of one of M, $\mathcal{M}(G)$ and is independent in the other.

Now for each $e \in E$ and each $v \in V$, $\partial_G(\{v\})$ is a union of cocircuits of M by hypothesis, and so $\partial_{G \setminus e}(\{v\})$ is a union of cocircuits of $M \setminus e$; because if D is a cocircuit of M then $D-\{e\}$ is a union of cocircuits of $M \setminus e$. However, if $e \notin X$ then X is independent in just one of $M \setminus e$, $\mathcal{M}(G \setminus e)$, and so $M \setminus e \neq \mathcal{M}(G \setminus e)$. By induction, we have

$$r(M \setminus e) > r(\mathcal{M}(G \setminus e)).$$

Thus each $e \in E-X$ is an isthmus of G but not a coloop of M, since by hypothesis $r(M) \leq r(\mathcal{M}(G))$. This has two consequences:

(a) No star of G has cardinality 1. For if $\partial(\{v\})=\{e\}$ say, then e is a coloop of M, since $\partial(\{v\})$ is a union of cocircuits of M. Hence $e \in X$; but that is impossible, because X is a circuit of one of M, $\mathcal{M}(G)$.

(b) G has just one circuit, namely X. For $E \neq \emptyset$, because $X \neq \emptyset$, and so by (a), G has a circuit. Each $e \in E-X$ is an isthmus of G, and so X includes all circuits of G. Thus X is not independent in $\mathcal{M}(G)$, and so X is a circuit of $\mathcal{M}(G)$, and independent in M.

From (a) and (b) it follows that G consists of a circuit together (possibly) with isolated vertices, and that $E=X$. Hence E is independent in M, and so

$$r(M) = |E| > r(\mathcal{M}(G))$$

contrary to hypothesis. This choice of X is therefore impossible, and so $M=\mathcal{M}(G)$, as required.

Corollary. *Let $G=(V, E)$ be a graph, and let M be a matroid on E. Then $M=\mathcal{M}(G)$ if and only if $r(M)=r(\mathcal{M}(G))$ and for each $v \in V$, every minimal nonempty cut of G included in $\partial(\{v\})$ is a cocircuit of M.*

This follows easily from the theorem when we observe that the minimal non-empty cuts of G included in $\partial(\{v\})$ form a partition of $\partial(\{v\})$.

It is known [2] that if M, M' are matroids on the same set, such that every cocircuit of M' is a union of cocircuits of M (that is, M' is a *strong image* of M) then $r(M) \geq r(M')$, and if equality occurs then $M=M'$. That suggests the following plausible strengthening of our theorem, which is regrettably false, however.

If M is a matroid on $E(G)$, then $\mathcal{M}(G)$ is a strong image of M if and only if every star of G is a union of cocircuits of M.

Counterexample. Let G be the graph with four vertices v_1, v_2, v_3, v_4 and five edges, two linking v_1, v_2, two linking v_3, v_4 and one linking v_1, v_3. Let M be the matroid on $E(G)$ in which $E(G)$ is a circuit. Then every star of G is a union of cocircuits of M, but the isthmus of G is not a coloop of M, and so $\mathcal{M}(G)$ is not a strong image of M.

2. Algorithms

Algorithms are well-known [1, 6] to test if a binary matroid is graphic (that is, is the polygon matroid of some graph) and if so to find such a graph. Thus to test if a general matroid is graphic, the obvious method is to test first if it is binary; if it is, we proceed using the established techniques for binary matroids, and if not, then it is not graphic. However, this is a bad method, because it is hard to test if a matroid is binary. We give another method which avoids this difficulty.

We assume as usual (see [3, 4, 5] for similar arguments) that the matroid we are concerned with is presented by means of an independent oracle; that is, we assume that we are able to determine whether any subset of $E(M)$ is independent, in unit time. The algorithm is as follows.

Pick a basis B. For each $x \in E(M)-B$, find the unique circuit C_x included in $B \cup \{x\}$. Let M' be the binary matroid on $E(M)$ in which B is a basis and each C_x is a circuit. Determine if M' is graphic (using e.g. the methods of [1]). If not, then M is not graphic. If M' is graphic, find a graph G such that $M'=\mathcal{M}(G)$. For each $v \in V(G)$, test if each minimal nonempty cut included in $\partial(\{v\})$ is a cocircuit of M. If so then M is graphic, and if not then M is not graphic.

It is easy to see that all these steps can be carried out in a time bounded above by a polynomial function of $|E(M)|$. The proof that the algorithm works is as follows. If M is binary then $M=M'$, and so if M' is not graphic, then either $M=M'$ and is not graphic, or $M \neq M'$ and is not binary and hence not graphic. Thus if M' is not graphic then neither is M. If M' is graphic and $M'=\mathcal{M}(G)$, then $M=\mathcal{M}(G)$ if and only if M is graphic. For if $M=\mathcal{M}(G)$ then obviously M is graphic, and conversely if M is graphic then it is binary and so $M=M'=\mathcal{M}(G)$. Thus it remains to test if $M=\mathcal{M}(G)$. We do this using the corollary to the theorem of section 1. The condition that $r(M)=r(\mathcal{M}(G))$ is automatically satisfied by the way M' is constructed.

We claimed earlier that there is no polynomially bounded algorithm to test if a matroid presented by an independence oracle is binary. We now prove that claim.

Choose a $2k$-element set $E=\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$; let $\mathcal{A}$ be the collection of all sets $\{x_i, y_i, x_j, y_j\}$ $(1 \leqq i<j \leqq k)$; and let $\mathcal{B}$ be the collection of all sets $\{z_1, \ldots, z_k\}$ where $z_i \in \{x_i, y_i\}$ $(1 \leqq i \leqq k)$ and $|\{z_1, \ldots, z_k\} \cap \{y_1, \ldots, y_k\}|$ is even. It is easy to see that when $k \geqq 3$, $\mathcal{A} \cup \mathcal{B}$ is the collection of circuits of a binary matroid on E. Now, for any $Z \in \mathcal{B}$, we can define a new matroid M_Z on E, in which Z is independent but every other subset of E is independent only if it is independent in M. (The reader may verify that this does define a matroid; indeed, if Z is a circuit and a hyperplane of any matroid, we can declare Z independent and leave all other sets unchanged, and this always makes a new matroid.) But if we have an algorithm to test if M is binary, it must ask the oracle "is Z independent?" for otherwise it will not distinguish between M and M_Z, which is not binary. Thus there are at least $|\mathcal{B}|$ demands made of the oracle, and so the algorithm takes a time at least

$$|\mathcal{B}| = 2^{k-1} = 2^{1/2|E(M)|-1}.$$

The time is therefore not bounded above by any polynomial function of $|E(M)|$, because this construction is possible for all $k \geqq 3$.

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